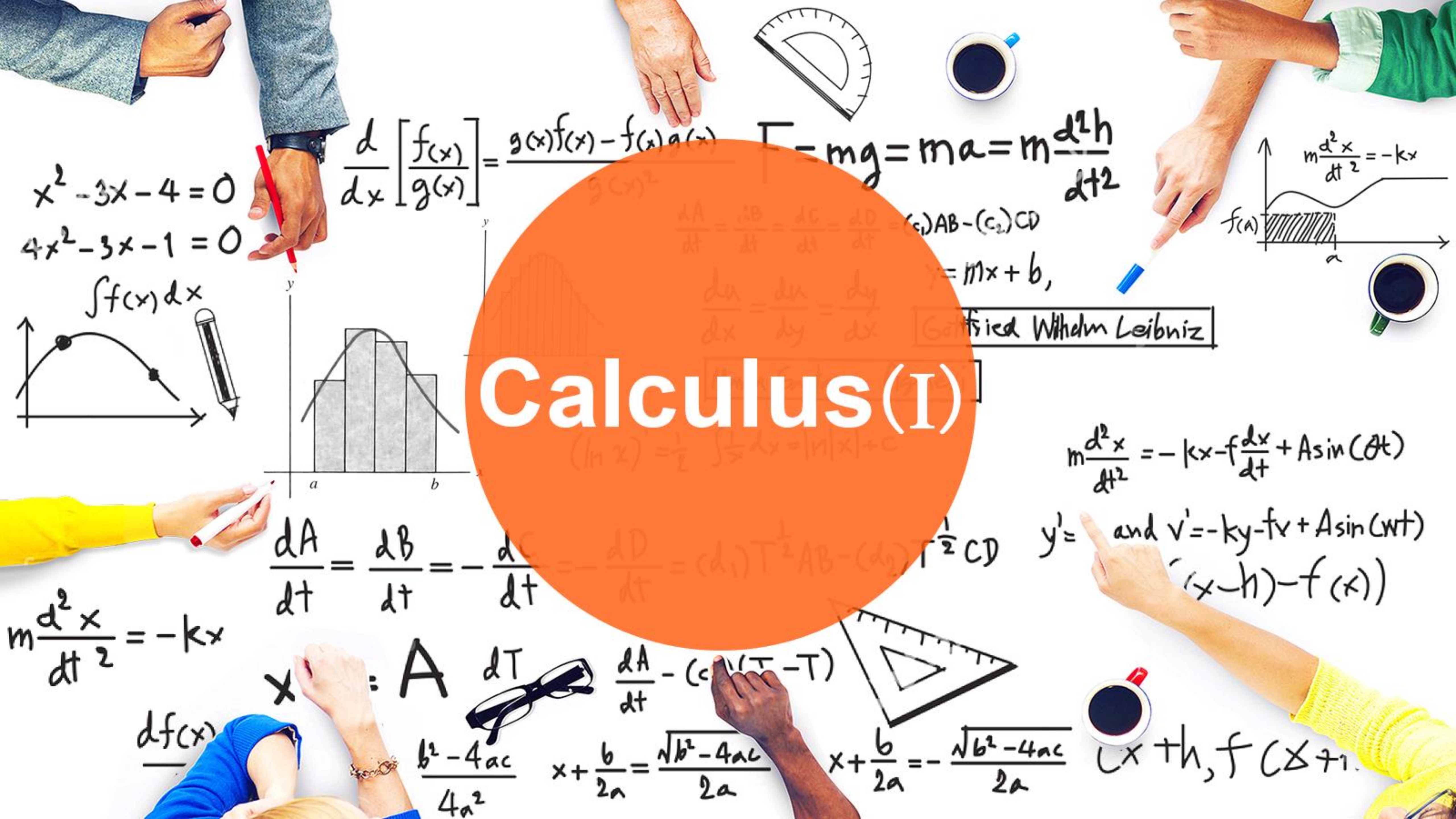
Calculus(I)


x² -3x - 4 = 0
4x² - 3x - 1 = 0
∫f(x) dx
m d²x/dt² = -kx
y' = and v' = -ky - fv + Asin(wt)
x + b/2a = √b² - 4ac / 2a
x + b/2a = - √b² - 4ac / 2a
Wilhelm Leibniz

4.2 The Definite Integral

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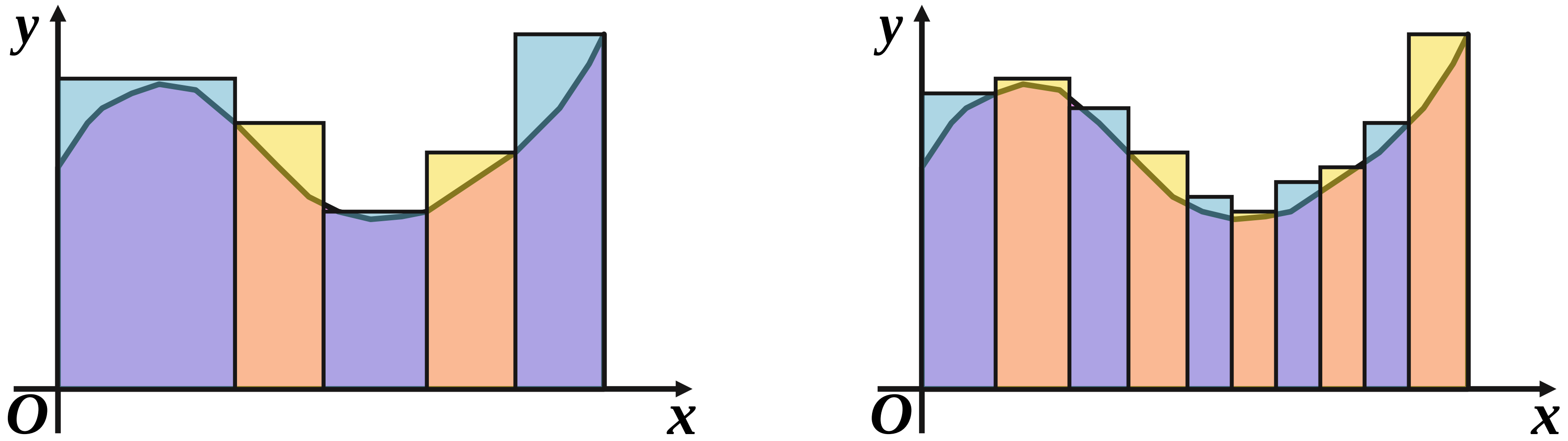
How to compute the area of the curve trapezoid?

Idea: $f(x) \equiv h$, rectangular area formula: $A = (b-a)h$ (base $\times$ height)

An: Rectangle area is approximate to replace the curve trapezoid area.

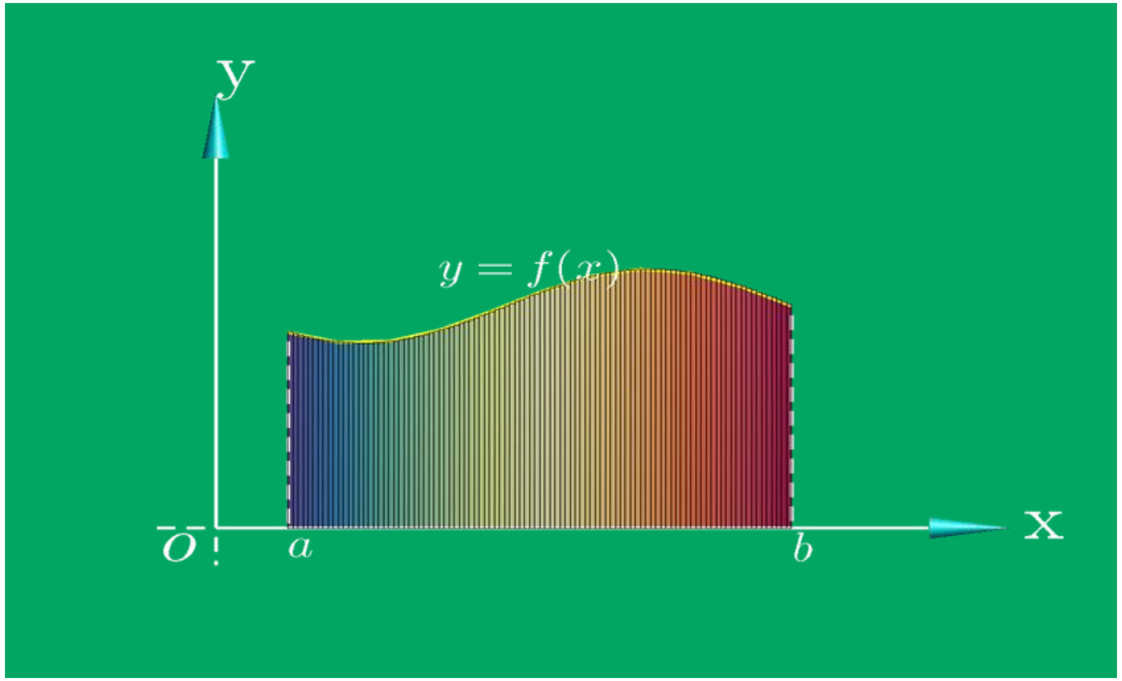


（ Five small rectangles ）　　（ Ten small rectangles ）

The Definition of Definite Integral

If we divide the curve trapezoid into a number of small rectangles,

when rectangle's width gradually reduces, then how to find the area of A?



The Definition of Definite Integral

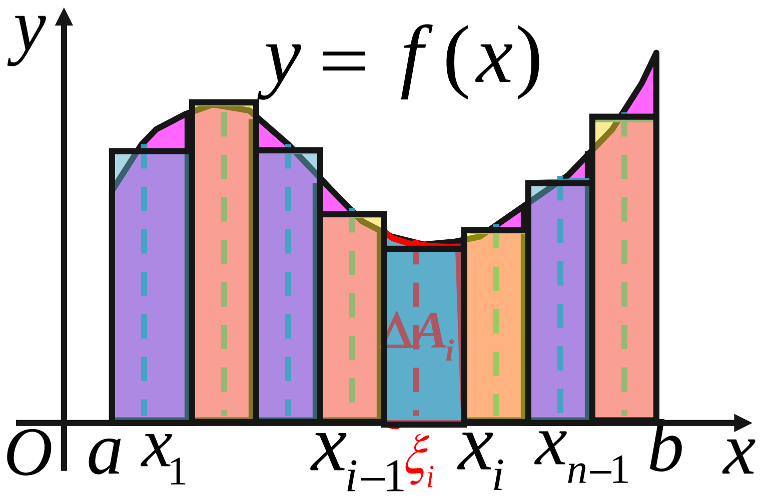
Step1: Segmentation $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,\ \Delta x_i = x_i - x_{i-1};$

Step2: Approximation $\Delta A_i \approx f(\xi_i)\Delta x_i\ (x_{i-1} \le \xi_i \le x_i);$

Step3: Sum $A \approx \sum_{i=1}^{n} f(\xi_i)\Delta x_i;$

Step4: Limit $A = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$

$$\lambda = \max\{\Delta x_1, \Delta x_2, \cdots \Delta x_n\}.$$



The Definition of Definite Integral

If the region A is bounded by $y = f(x) > 0,\quad x = a, x = b$ and $y = 0$,

and the area of A denoted as I, we have

$$\int_a^b f(x)\,\mathrm{d}x = I = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

upper limit

integral variable

integral sum

lower limit

integral function

Supplement

Finally, we point out that x is a dummy variable in the symbol $\int_a^b f(x)dx$.

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(u)du$$

Theorem A

Integrability Theorem

If f is bounded on $[a, b]$ and if it is continuous there except at a finite number of points, then f is integrable on $[a, b]$.

In particular, if f is continuous on the whole interval $[a, b]$, it is integrable on $[a, b]$.

1. Polynomial functions

2. Sine and cosine functions

3. Rational functions, provided that the interval $[a, b]$ contains no points where the denominator is 0

Theorem B

Interval Additive Property

If f is integrable on an interval containing the points a, b, and c, then

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

no matter what the order of a, b, and c.

Example 1

Find the area of the region bounded by $y = x^2$, $x = 1$ and x-axis. $\int_0^1 x^2 \, dx$

Let $0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1$,

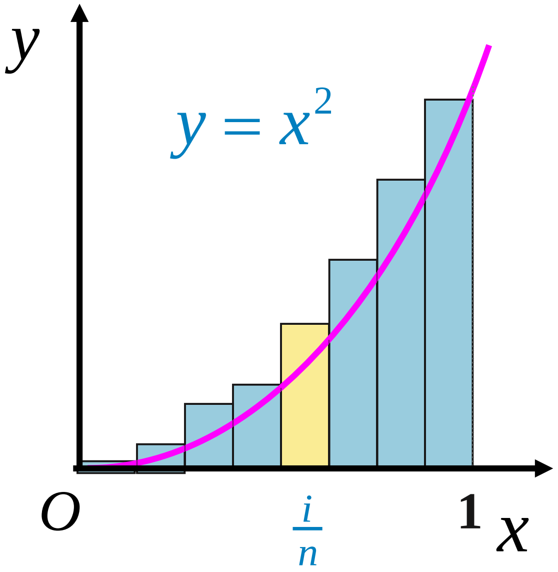
$x_i = \frac{i}{n}$, $\Delta x_i = \frac{1}{n}$, $i = 1, 2, \cdots, n$,

we let, $\xi_i = x_i$, $i = 1, 2 \cdots n$,thus,

$$\sum_{i=1}^{n} f(\xi_i)\Delta x_i = \sum_{i=1}^{n} \xi_i^2 \Delta x_i = \sum_{i=1}^{n} x_i^2 \Delta x_i$$

($x_i = \frac{i}{n}$, $\Delta x_i = \frac{1}{n}$)

$$= \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{i=1}^{n} i^2$$



Example 1

$$= \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \quad \lambda \to 0 \ \Rightarrow \ n \to \infty$$

$$\int_0^1 x^2 \mathrm{d}x = \lim_{\lambda\to 0}\sum_{i=1}^{n} \xi_i^2 \Delta x_i = \lim_{n\to\infty}\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \boxed{= \frac{1}{3}.}$$

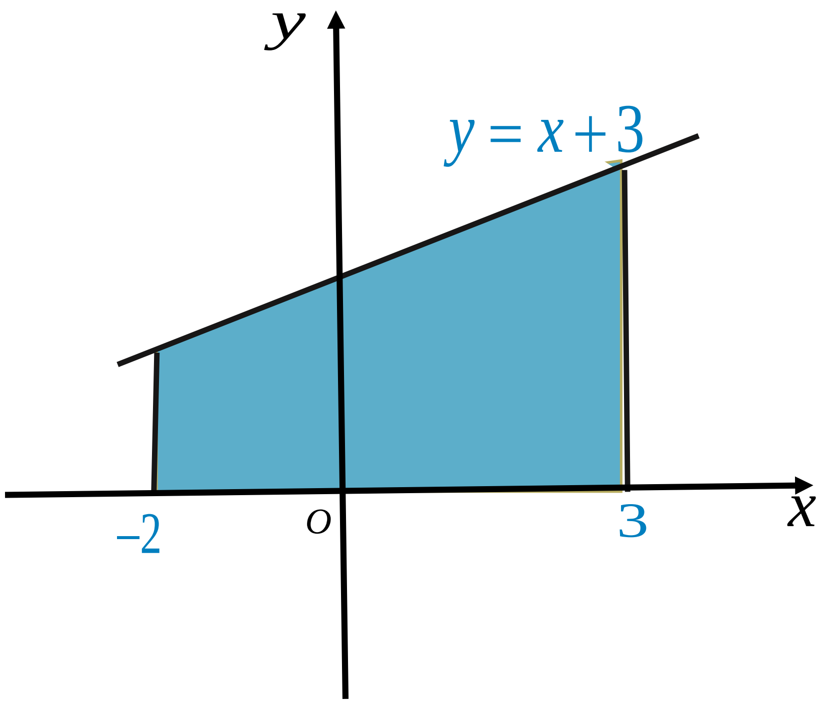
$$\int_0^1 x^2 \mathrm{d}x \approx \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

When n takes different values, $\int_0^1 x^2 \mathrm{d}x$ has different approximation accuracy.

Example 2

Evaluate $\int_{-2}^{3}(x+3)dx$ by using a regular partition.

$\Delta x = \frac{5}{n}$, $\xi_i = x_i$ as the sample point,



$$\begin{cases} x_0 = -2 \\ x_1 = -2 + \frac{5}{n} \\ x_2 = -2 + 2(\frac{5}{n}) \\ \cdots \\ x_i = -2 + i(\frac{5}{n}) \\ \cdots \\ x_n = -2 + n(\frac{5}{n}) = 3 \end{cases}$$

Example 2

So, we have $f(\xi_i) = f(x_i) = x_i + 3 = 1 + i(\frac{5}{n})$

$$\sum_{i=1}^{n} f(\xi_i)\Delta x_i = \sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{n}[1 + i(\frac{5}{n})].\frac{5}{n}$$

$$= 5 + \frac{25}{2} + \frac{25}{2n},$$

$$\int_{-2}^{3}(x+3)dx = \lim_{\lambda \to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i \quad (n \to \infty)$$

$$= \frac{35}{2}.$$

Summary of the Definite Integral

(1) When $a = b$, $\int_a^b f(x)\mathrm{d}x = 0$

(2) When $a > b$, $\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x$

If a function has definite integral, thus we don't consider the values of a and b which is bigger.

(1) $\int_a^b [f(x) \pm g(x)]\mathrm{d}x = \int_a^b f(x)\mathrm{d}x \pm \int_a^b g(x)\mathrm{d}x$ Linear Characteristics

(2) $\int_a^b kf(x)\mathrm{d}x = k\int_a^b f(x)\mathrm{d}x$ (k is a constant number).

Summary of the Definite Integral

(3) Let $a < c < b$

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$$

Interval additivity

(4) $\int_a^b 1dx = b - a$

(5) $f(x) \geq 0$, so $\int_a^b f(x)dx \geq 0$

Sign preserving

Questions and Answers

Find the limit: $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right)$

$$\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\cdots+\frac{1}{1+\frac{n}{n}}\right)\frac{1}{n}$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{1+\frac{1}{i}}\right)\frac{1}{n}$$

$$=\int_0^1\frac{1}{1+x}dx=\ln(1+x)\Big|_0^1$$

$$=\ln 2.$$

Questions and Answers

Find the limit: $\lim_{n\to\infty}\left(\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-4}}+\cdots+\frac{1}{\sqrt{4n^2-n^2}}\right)$

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-4}}+\cdots+\frac{1}{\sqrt{4n^2-n^2}}\right)$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{4-(1/n)^2}}+\frac{1}{\sqrt{4-(2/n)^2}}+\cdots+\frac{1}{\sqrt{4-(n/n)^2}}\right)\frac{1}{n}$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{4-(1/i)^2}}\right)\frac{1}{n}=\int_0^1\frac{1}{\sqrt{4-x^2}}dx$$

$$=\frac{\pi}{6}.$$

The Definite Integral